4^{th} Iranian Combinatorics Olympiad

31^{th} October & 1^{st} November, 2024

Contest problems with solutions

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This booklet is prepared by Afrouz Jabal Ameli and Faezeh Motiei.

Participating Countries

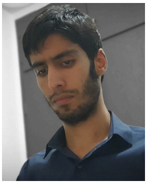
List of countries that participated in the 4^{th} Iranian Combinatorics Olympiad

Afghanistan	Albania	Armenia	Austria
Azerbaijan	Bangladesh	Belarus	Bolivia
Bosnia	Bulgaria	Canada	Chile
Croatia	Cuba	Czech Republic	Dominican Republic
Ecuador	Egypt	England	France
Georgia	Hong Kong	India	Indonesia
Iran	Iraq	Ireland	Italy
Jamaica	Jordan	Kazakhstan	Kyrgyzstan
Liechtenstein	Malaysia	Moldova	Mongolia
Myanmar	Nepal	Netherlands	Nicaragua
North Macedonia	Pakistan	Panama	Peru
Poland	Romania	Saudi Arabia	Serbia
Singapore	Slovakia	Slovenia	Somalia
Switzerland	Syria	Taiwan	Tunisia
Turkey	Turkmenistan	United States	Uruguay
Uzbekistan	Venezuela	Vietnam	Zimbabwe


الله اكبر

The 4th Iranian Combinatorics Olympiad was held on October 31 and November 1, 2024, bringing together over 2,900 participants in nearly 1,200 teams from 64 countries. The Problem Selection Committee for this contest consisted of

Alireza
Alipour



Abolfazl
Asadi

Mehdi
Haji Beigi

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Hosseini

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Motiei

Morteza
Saghafian

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First Round

Problems

Problem 1.

Morteza and Joseph have 100 boxes such that for each $1 \leq i \leq 100$ one of these boxes has exactly i coins. Joseph puts the boxes on top of each other in an order that he prefers. Now in 10 steps, Morteza collects coins in the following way:

> at each step, Morteza picks up the top 10 boxes, opens them, and then selects one of the boxes that he has already opened (this also includes all the boxes that he has opened in the previous steps) and collects all the coins of that box.

What is the maximum number of coins that Morteza can always collect regardless of the way Joseph orders the boxes?

(→ p.9)

Problem 2.

Let S be a set of six points in the plane such that no three points of S are collinear. What is the maximum number of ways that we can partition these 6 points into two subsets of size three such that the sides of the triangle formed by the points in each group do not intersect the sides of the triangle formed by the other group?

(→ p.9)

Problem 3.

We say that a sequence $x_1, x_2, \ldots, x_n$ is increasing if $x_i \leq x_{i+1}$ for all $1 \leq i < n$. How many ways are there to fill an 8×8 table by numbers 1, 2, 3, and 4 such that:

- The numbers in each row are increasing from left to right,
- The numbers in each column are increasing from top to bottom,
- and there is no pair of adjacent cells such that one is filled with 2 and the other one is filled with 3. (We say two distinct cells are adjacent if they share a side)

(→ p.9)

Problem 4.

Matin and Morteza are playing a game together on a graph with 100 vertices. At the beginning of the game, the graph has no edges. In each turn, Matin picks a vertex that is not *full*, say u and Morteza must add a new edge that has u as one of its endpoints while keeping the graph simple (A vertex is full if it is adjacent to all other vertices of the graph). The game ends as soon as the graph has a cycle of even length. Matin wishes to maximize the number of edges, whereas Morteza wants to minimize this. Assuming both players play optimally, what is the number of edges of the final graph?

(→ p.9)

Problem 5.

We say that a coloring of the 7×7 table is nice if,

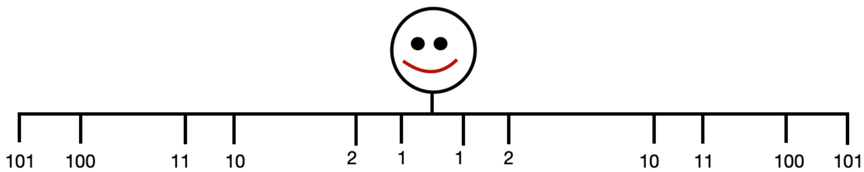
- Every cell is colored by blue or red,
- Every cell has precisely one diagonal neighbor that is red.

Determine the number of nice colorings.
(A cell A is a diagonal neighbor of cell B if A and B share exactly one point. For instance, each of the four cells on the corners of the table has one diagonal neighbor.)

(→ p.9)

Problem 6.

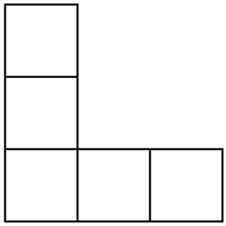
Below we have a figure of a spider named Johny who has 12 feet. The numbers in the figure correspond to the distance of Johny's feet from his head. For taking a nap on his web, Johny doesn't need to hold all his feet down but he must keep his balance and that means, (1) he should hold at least one foot down on the web, and (2) the total distances of his left feet that are on the web from his head should be equal to the total distances of his right feet that are on the web from his head. Determine the number of ways that Johny can take a nap.



(→ p.9)

Problem 7.

In a 7×7 table initially, all the cells are white. At each step, we can identify a tile as depicted in the figure below (with rotations allowed) on the table and flip the colors (from white to black and black to white) of all those five cells in the table simultaneously. After finitely many steps, what is the maximum possible number of black cells in the table?



(→ p.9)

Problem 8.

Given a sequence $S =< a_1, \dots, a_n >$ of integers, the number of inversions of S is equal to the number of pairs $1 \leq i < j \leq n$ such that $a_i > a_j$. Let $Z =< a_1, \dots, a_{20} >$ be a sequence of 20 elements from $\{1, \dots, 10\}$, where $a_1, \dots, a_{10}$ is a permutation of $\{1, \dots, 10\}$ and for every $1 \leq i \leq 10$, $a_{10+i} = 11 - a_i$. Let A and B be the minimum and maximum possible values for the inversion number of Z, respectively. What is A+B?

(→ p.9)

Problem 9.

Rostam wants to find a sequence of numbers $< a_1, ..., a_{2025} >$ such that $0 \leq a_i \leq 1023$ and if you place them around a circle in the same order (as they appear in the sequence), then each number is **XOR** of its two neighbors on the circle. How many ways are there to do this?
The XOR of two non-negative integers x and y is defined in the following way: Assume $(\overline{x_k...x_1})_2$ and $(\overline{y_t...y_1})_2$ are their corresponding binary representations and without loss of generality, assume $k \geq t$. Now, let $y_{t+1} = \cdots = y_k = 0$.
Then, the binary representation of their XOR is $(\overline{z_1...z_k})_2$ where $z_i = 0$ if $x_i = y_i$ and $z_i = 1$ if $x_i \neq y_i$.
For instance, let $x = 49$ and $y = 101$, then $x = (110001)_2$ and $y = (1100101)$. Now, the binary representation of their XOR is $(1010100)_2$ which means their XOR is 84.

(→ p.9)

Problem 10.

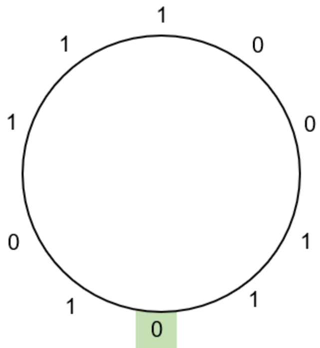
Bahman has an 8×8 table. Originally, all the cells are empty and white. He fills each of the cells with a number from 1 to 4. Then, he chooses two numbers a and b such that $1 \leq a < b \leq 4$ and colors all the cells that are filled with a or b in gray. The value of the table is $64 \times t$, where t is the number of rows that are completely gray after this process. For instance, in the table below, if we choose $i = 2$ and $j = 3$, we would have $t = 3$ rows that are completely gray, and hence the value of the table is 192. What is the average value of the table over all the possible ways that Bahman can fill the table and choose a and b?

1	2	1	3	2	3	3	4
1	3	3	3	2	1	2	1
4	2	4	1	4	3	4	4
2	2	2	3	3	2	3	2
1	2	3	4	1	2	3	4
2	2	2	2	2	2	2	2
3	2	3	3	3	2	3	2
3	4	4	2	2	1	1	1

(→ p.9)

Problem 11.

There are 10 cells around a circle and in each cell, there is a digit which is either one or zero. The process of creating a string from a given cell C is to start from C and an empty string S and then in 10 steps, we repeat the following: Move to one of the two neighboring cells and append the digit inside that cell to the end (right) of S.
Assume from <u>all the cells</u> we are able to create $S = 0101010101$. How many different ways are there to initialize the values of all the cells with zero or one such that this is possible?
For instance, in the circle below, if we start from the colored cell, we cannot create $S = 0101010101$ but we can create $S = 1010101010$. Two ways of initializing the values of the cells such as A and B are the same if you can reach A by shifting B around the circle.



(→ p.9)

Problem 12.

Assume S_n is the set of all ordered n-tuples of 0 and 1 and let $A_1, A_2, \dots, A_{32}$ be a permutation of the elements of S_5. Also assume that $f(A_1) = 1$ and for every $1 \le i \le 32$ the value of $f(A_i)$ is equal to the smallest positive integer such that for every j $(1 \le j < i)$, where A_i and A_j differ in exactly one coordinate, it holds that $f(A_i) \ne f(A_j)$ (For instance, if $A_i = (0, 1, 1, 0, 1)$ and $A_j = (0, 1, 0, 0, 1)$ then A_i and A_j differ only in the third coordinate). What is the maximum possible value of $\max\{f(A_1), f(A_2), \dots, f(A_{32})\}$.

(→ p.9)

Switching

A graph is a *tree* if it is connected and does not contain any cycles. Let T be a tree. The *diameter* of T is defined as the length (the number of edges) of the longest path of T. At each step, we are allowed to update T by removing an edge and then drawing a new edge such that T remains a tree.

Problem 13.

If T has 2024 vertices and is of diameter 100, after one step what is the minimum possible diameter of T?

(→ p.9)

Problem 14.

If T has 2024 vertices and is of diameter 100, after one step what is the maximum possible diameter of T?

(→ p.9)

Problem 15.

What is the minimum possible number k, such that for every tree T with 2024 vertices and diameter 100, one can reach a tree of a diameter smaller than 100 in at most k steps?

(→ p.9)

Answers

Problem	**Answer**
1	550
2	7
3	165636900
4	101
5	192
6	123
7	48
8	180
9	1048576
10	2
11	1
12	6
13	51
14	199
15	39

Second Round

Problems

Problem 1.

Consider a 13×13 table in which the rows are numbered from top to bottom and the columns are numbered from left to right by numbers 1 to 13. Assume that the cells that are intersections of an even row with an even column are blue and the other cells are red (There are 36 blue cells in the table). What is the maximum number of 1×3 and 3×1 tiles that we can place inside this table such that (1) each rectangle only contains red cells and (2) no cell belongs to more than one of these rectangles?

(→ p.17)

Problem 2.

We are given a permutation of $\{1, 2, ..., n\}$, say $\Pi =< \pi_1, ..., \pi_n >$. The swapping step is defined as follows:

- Choose two number s and t, such that $1 \leq s < t \leq n$ and $t - s$ is odd.
- For each $0 \leq i < \frac{t-s+1}{2}$, update Π by swapping π_{s+2i} and π_{s+2i+1}.

For instance, applying a swapping step with $s = 3$ and $t = 8$ on the permutation $< 10, 9, 8, 7, 6, 5, 4, 3, 2, 1 >$ results in the permutation $< 10, 9, 7, 8, 5, 6, 3, 4, 2, 1 >$. Assume that $n \geq 3$ and we start with $\Pi =< n, n-1, \ldots, 1 >$.

a) Prove that at least n steps is required to modify Π to $< 1, 2, ..., n >$.

b) Prove that it is possible to modify Π to $< 1, 2, ..., n >$ in exactly n steps.

(→ p.19)

Problem 3.

Let $A_1, \ldots, A_{20}$ be subsets of size three of the set $X = \{1, 2, \ldots, 10\}$. We say that a subset S of X is a covering subset if for every $1 \leq i \leq 20$, it holds that $S \cap A_i \neq \emptyset$. What is the minimum possible value of k, such that a covering subset of size k always exists?

($\rightarrow$ p.20)

Problem 4.

We call a graph *planar* if it can be drawn on the plane so that its edges intersect only at their endpoints. Let $n \geq 3$ be an integer and G be a planar graph on n vertices. Determine the maximum possible number of cycles of length 3 of G in terms of n.

($\rightarrow$ p.21)

Problem 5.

Let G be a simple graph and let V be the set of vertices of G. We denote by $f(G)$ the maximum number k such that there exists a subset $S \subseteq V$ with $|S| = k$, in which every vertex in S has at most one neighbor in S.

a) Compute $f(G)$ if $V = \{0, 1, \ldots, n-1\}$ and $E = \{uv | u - v \equiv r \pmod{n}$ where $r \in \{-2, -1, 1, 2\}\}$.

b) Assume each vertex $v \in V$ corresponds to a unique sequence of length n consisting of elements in $\{0, 1, 2\}$ ($|V(G)| = 3^n$). A pair of vertices are adjacent in G if their corresponding sequences differ in exactly one element (one bit). Prove that $f(G) > 3^{n-1}$. For example, if $n = 4$ the vertices corresponding to sequence $< 0, 1, 2, 0 >$ and $< 0, 1, 1, 0 >$ only differ in the third position and hence they are adjacent. Prove that $f(G) > 3^{n-1}$.

($\rightarrow$ p.23)

Problem 6.

Consider the increasing sequence $1, 2, \ldots, n$. In each move, we first take two adjacent elements x and y that are still positive, and then by spending $\min\{x, y\}$ many gold coins we decrease both elements by $\min\{x, y\}$. What is the minimum number of gold coins that we have to spend in order to reach a sequence in which no more moves are possible?

($\rightarrow$ p.25)

Problem 7.

There are 100 points on the plane, namely $P_1, \ldots, P_{100}$ such that no three points are collinear. Assume for every i and j such that $1 \leq i < j - 1 \leq 98$, the segments P_iP_{i+1}, P_jP_{j+1} do not intersect (i.e., do not share a common point). We say that $\{P_i, P_{i+1}, P_{i+2}\}$ is an *empty* consecutive triple if the triangle $\triangle P_iP_{i+1}P_{i+2}$ does not contain any of the other points. Find the largest number k such that one can always find at least k empty consecutive triples.

(→ p.26)

Solutions

Problem 1.

Consider a 13×13 table in which the rows are numbered from top to bottom and the columns are numbered from left to right by numbers 1 to 13. Assume that the cells that are intersections of an even row with an even column are blue and the other cells are red (There are 36 blue cells in the table). What is the maximum number of 1×3 and 3×1 tiles that we can place inside this table such that (1) each rectangle only contains red cells and (2) no cell belongs to more than one of these rectangles?

Proposed by Alireza Alipour

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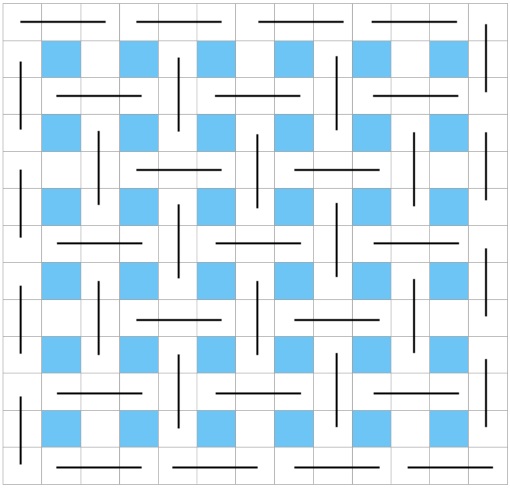
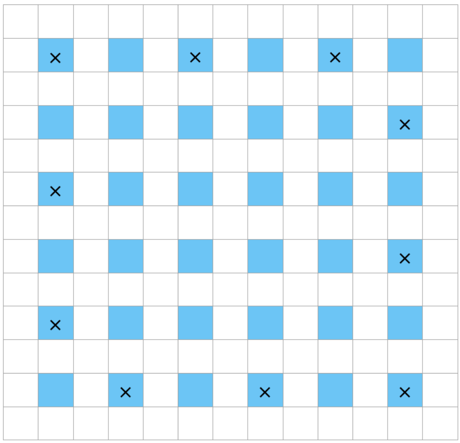
Solution.

- The answer is 41.
- The first figure shows a way to place 41 rectangles that satisfy the conditions of the problem.



- Now one can easily show that for each of the 10 marked blue cells in the second figure, say for a marked cell C, it is not possible to place the rectangles in a way that all the neighbors of C are covered at the same time (Two cells are neighbors if they share a side).



Therefore one can place at most $\frac{169-36-10}{3} = 41$ cells in this table.

■

Problem 2.

We are given a permutation of $\{1, 2, ..., n\}$, say $\Pi =< \pi_1, ..., \pi_n >$. The swapping step is defined as follows:

- Choose two number s and t, such that $1 \leq s < t \leq n$ and $t - s$ is odd.
- For each $0 \leq i < \frac{t-s+1}{2}$, update Π by swapping π_{s+2i} and π_{s+2i+1}.

For instance, applying a swapping step with $s = 3$ and $t = 8$ on the permutation $< 10, 9, 8, 7, 6, 5, 4, 3, 2, 1 >$ results in the permutation $< 10, 9, 7, 8, 5, 6, 3, 4, 2, 1 >$. Assume that $n \geq 3$ and we start with $\Pi =< n, n-1, \ldots, 1 >$.

a) Prove that at least n steps is required to modify Π to $< 1, 2, ..., n >$.

b) Prove that it is possible to modify Π to $< 1, 2, ..., n >$ in exactly n steps.

Proposed by Mehdi Haji Beigi

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Solution.

a) Note that we have to move each of n and 1, $n-1$ units from their initial spot. Therefore if one is able to sort the sequence in $n-1$ many steps, then both 1 and n must be moved towards their final spot at all the steps. If n is odd this is impossible, as in the first step one can not move these two at the same time a contradiction. If n is even, at the first step we must set $s = 1$ and $t = n$. After the first step $\pi_n = 2$ and hence 2 is $n-2$ units away from its final spot, so at all the remaining steps we must move 1 and 2 together towards the left of the sequence. But this is not possible already in the second step, a contradiction.

b) If n is odd, for every $i \in \{1, 3, 5, \ldots, n\}$, at step i we set $s = 1$ and $t = n-1$, and for every $i \in \{1, 3, 5, \ldots, n\}$, at step i we set $s = 2$ and $t = n$.

If n is even, for every $i \in \{1, 3, 5, \ldots, n-1\}$, at step i we set $s = 1$ and $t = n$, and for every $i \in \{2, 4, 6, \ldots, n-1\}$, at step i we set $s = 2$ and $t = n-1$.

■

Problem 3.

Let $A_1, \ldots, A_{20}$ be subsets of size three of the set $X = \{1, 2, \ldots, 10\}$. We say that a subset S of X is a covering subset if for every $1 \leq i \leq 20$, it holds that $S \cap A_i \neq \emptyset$. What is the minimum possible value of k, such that a covering subset of size k always exists?

Proposed by Alireza Alipour

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Solution.

- The answer is 6.
- If $A_1, \ldots, A_{10}$ is the set of all subsets of size 3 of $B = \{1, \ldots, 5\}$ and $A_{11}, \ldots, A_{20}$ is the set of all subsets of size 3 of $C = \{6, \ldots, 10\}$, then any covering subset must have at least 3 elements in A and 3 elements in B. Hence, in this case, the size of the minimum covering set is at least 6.
- To show that the size of the minimum covering set is always at most 6, we provide two solutions:
  - Proof: If there is a subset B of X with $|B| = 4$ such that for every i, $A_i \not\subset B$ then, $X \setminus B$ is a covering subset of size 6, and hence the claim. Thus assume for every $B \subset X$ with $|B| = 4$, this does not hold. Note that for every A_i there are exactly $10 - 3 = 7$ subsets B of X with $|B| = 4$ and $A_i \subset B$. This means that $7 \times 20 \geq \frac{10!}{4!6!} = 210$, a contradiction!
  - Alternative Proof: We produce a covering set S of size 6 by applying a greedy strategy. We start by $S = \emptyset$ and the assumption that non of the sets A_1, A_2, $\ldots$, A_{20} is covered yet. At each stage take the element that belongs to the maximum number of uncovered subsets, say s, add s to S, and consider the uncovered subsets that contain s as covered sets. We terminate the process as soon as all the subsets are covered.
    Using the pigeonhole principle one can show that after the first stage, the number of uncovered sets is at most $20 - \frac{20 \times 3}{10} = 14$, after the second stage this becomes at most $14 - ([\frac{14 \times 3}{9}] + 1) = 9$, then it becomes at most $9 - ([\frac{9 \times 3}{8}] + 1) = 5$, then it becomes at most $5 - ([\frac{5 \times 3}{7}] + 1) = 2$, then it becomes 1 and at the last stage we can cover the (possible) final uncovered set.

■

Problem 4.

We call a graph *planar* if it can be drawn on the plane so that its edges intersect only at their endpoints. Let $n \geq 3$ be an integer and G be a planar graph on n vertices. Determine the maximum possible number of cycles of length 3 of G in terms of n.

Proposed by Morteza Saghafian

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Solution.

- The answer is $3n - 8$.
- First, we provide a planar graph G_n on n vertices with $3n - 8$ triangles for every n and it contains an inner face that is a triangle and has no other vertices inside this face. For $n = 3$ we set $G_3 := K_3$. Now assume for some n, G_n exists. We construct G_{n+1} from G_n by adding a new vertex v inside an empty triangle face F of G_n and then we connect v to all the three vertices of F. Note that after this we introduce three new triangles, that are empty faces and hence the claim.
- To show that every planar graph has at most $3n-8$ triangles, we provide two solutions:
  - Solution 1: We use induction on n. The condition holds for $n = 3$. Assume the inductive hypothesis holds for 1 to n. Now let G be a planar graph on $n+1$ vertices and consider a fixed planar drawing of G. If all the triangles of G are faces of G, then by applying the Euler formula we have an upper bound of $2n - 5$ on the number of faces and hence the same bound on the number of vertices. This completes the argument as $2n - 5 \leq 3n - 8$. So let us assume there exists a triangle T, that is not a face of G. This means that there is at least one vertex inside T, as T is not an inner face and also there is at least one vertex outside T as T is not an outer face. Let k be the number of other vertices inside T. Therefore there are $n - k - 3$ vertices outside T. By applying induction there are at most $3(k+3) - 8$ triangles that include vertices of T and the points inside T and there are at most $3(n-k) - 8$ triangles that include vertices of T and the points outside T. Since T has been taken into account twice, then the total number of triangles is at most $3(k+3) - 8 + 3(n-k) - 8 - 1 = 3n - 8$.

– Solution 2: First we show that we can assume all the vertices have a degree at least 2. This can be guaranteed iteratively by removing vertices of degree 0 or 1 until no such vertex exists anymore (this is possible as these vertices do not belong to any triangle and also their removal does not violate the planarity of the graph). Now let $v_1, \ldots, v_n$ be the vertices of the graph. For every $1 \leq i \leq n$, let d_i be the degree of v_i and let t_i be the number of triangles that have v_i as one of their vertices. Note that this means there are t_i edges in the induced subgraph of the neighbors of v_i. Therefore if we look at the induced subgraph of v_i and all its neighbors, this would be a planar graph with $d_i + 1$ vertices and $d_i + t_i$ edges. By applying the Euler formula, one has:

$$d_i + t_i \leq 3(d_i + 1) - 6,$$

and hence:

$$t_i \leq 2d_i - 3.$$

Therefore:

$$\begin{aligned}
\# \text{ Triangles} &= \sum_{i=1}^{n} \frac{t_i}{3} \\
&\leq \sum_{i=1}^{n} \frac{2d_i - 3}{3} \\
&= \frac{4|E(G)|}{3} - n \\
&\leq \frac{4}{3} \times (3n - 6) - n \\
&= 3n - 8.
\end{aligned}$$

■

Problem 5.

Let G be a simple graph and let V be the set of vertices of G. We denote by $f(G)$ the maximum number k such that there exists a subset $S \subseteq V$ with $|S| = k$, in which every vertex in S has at most one neighbor in S.

a) Compute $f(G)$ if $V = \{0, 1, \dots, n-1\}$ and $E = \{uv | u-v \equiv r \pmod{n}$ where $r \in \{-2, -1, 1, 2\}\}$.

b) Assume each vertex $v \in V$ corresponds to a unique sequence of length n consisting of elements in $\{0, 1, 2\}$ ($|V(G)| = 3^n$). A pair of vertices are adjacent in G if their corresponding sequences differ in exactly one element (one bit). Prove that $f(G) > 3^{n-1}$. For example, if $n = 4$ the vertices corresponding to sequence $< 0, 1, 2, 0 >$ and $< 0, 1, 1, 0 >$ only differ in the third position and hence they are adjacent. Prove that $f(G) > 3^{n-1}$.

Proposed by Alireza Alipour

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Solution.

a) Let us assume that $n \geq 4$ (For smaller values it is easy to calculate $f(G)$.)

Let us also assume that the vertices are arranged around a circle for 0 to n. We denote by S, a subset of size $f(G)$ that satisfies the conditions of the problem. Now observe that from every four consecutive vertices, at most two of them belong to S. This already suggests that $f(G) \leq [\frac{n}{2}]$. So if $n = 4k$ or $n = 4k + 1$ then $f(G) \leq 2k$ and if $n = 4k + 3$ then $f(G) \leq 2k + 1$.

Now for $n = 4k + 2$, we strengthen the upper-bound by showing $f(G) \leq [\frac{n}{2}] - 1 = 2k$. Note that by the above argument, S contains at most two elements from every four consecutive vertices, and if $f(G) = \frac{n}{2} = 2k+1$, then $S = \{0, 2, 4, \dots, 4k\}$ or $S = \{1, 3, 5, \dots, 4k + 1\}$. In both cases, S is not valid as all the vertices in S have two neighbors inside S. Therefore if $n = 4k + 2$ then $f(G) \leq 2k$.

We complete the argument by providing sets S that satisfy the properties of the problem and also attain our upper bound. Now for $n = 4k$, $4k+1$, and $4k+2$, the set $S := \{0, 1, 4, 5, 8, 9, 12, 13, \dots, 4k - 4, 4k - 3\}$ satisfies the property and has $2k$ elements. For $n = 4k + 3$, $S := \{0, 1, 4, 5, 8, 9, 12, 13, \dots, 4k - 4, 4k - 3\} \cup \{4k\}$ satisfies the property and has $2k + 1$ elements.

b) Let $A_n = \{(x_1, x_2, \ldots, x_n) | (x_1+x_2+\cdots+x_n) \equiv 0 \pmod 3\}$. We use induction on n to show that we can select $S_n \subseteq V$ such that $|S_n| > 3^{n-1}$, $S_n \cup A_n = \emptyset$, and S_n satisfies the properties in the problem statement. For $n = 1$, we set $S_1 = \{(1), (2)\}$. Now assume the inductive hypothesis holds for n. We set

$$S_{n+1} = \{(x_1, x_2, \ldots, x_n, 0) | (x_1, x_2, \ldots, x_n) \in S_n\} \bigcup$$

$$\{(y_1, y_2, \ldots, y_n, j) | 1 \leq j \leq 2 \text{ and } (y_1, y_2, \ldots, y_n) \in A_n\}.$$

One can show that every vertex in S_{n+1} is adjacent to at most one vertex in S_{n+1} as (1) if the last coordinate of a vertex in S_{n+1} is zero then by construction this vertex can only be adjacent to vertices in S_{n+1} whose last bit are also zero, and (2) if the last coordinate of a vertex $v \in S_{n+1}$ is $j \in \{2, 3\}$, then the only neighbor of this vertex in S_{n+1} is the vertex u, such that the last coordinate is $5 - j$ and its other coordinate are the same as v.

Furthermore, by construction the size of $|S_{n+1}| = |S_n| + 2|A_n|$. We know that $|S_n| > 3^{n-1}$. Also, we can show that $|A_n| = 3^{n-1}$, as there are 3^{n-1} as by choosing $n-1$ coordinates of an element in A_n in all the possible 3^{n-1} ways, for the last coordinate there is exactly one valid choice for the last coordinate. Altogether, we have:

$$|S_{n+1}| = |S_n| + 2|A_n| > 3^{n-1} + 2 \times 3^{n-1} = 3^n$$

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Problem 6.

Consider the increasing sequence $1, 2, \ldots, n$. In each move, we first take two adjacent elements x and y that are still positive, and then by spending $\min\{x, y\}$ many gold coins we decrease both elements by $\min\{x, y\}$. What is the minimum number of gold coins that we have to spend in order to reach a sequence in which no more moves are possible?

Proposed by Mehdi Haji Beigi

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Solution.

- The answer is $(n-1)+(n-4)+(n-7)+.\cdots+r$, where r is the smallest positive integer such that $n-1 \equiv 3 \pmod 3$ (i.e. r is the remainder of $n-1$ divided by 3).
- We apply induction on n to show that there is a way to do the moves such that in total we spend $(n-1)+(n-4)+(n-7)+.\cdots+r$ many gold coins. In the first step, we apply the move on $n-2$ and $n-1$. After this, the last three elements will be 0, 1, and n. Now if we apply it to the last two elements the sequence becomes $1, 2, 3, \ldots, n-3, 0, 0, n-1$. Note that so far we spent $n-1$ many gold coins. Now we can forget about the last three elements and by induction there is a way to deal with the first $n-3$ elements by spending at most $(n-4)+(n-7)+\cdots+r$ many gold coins and this completes the argument.
- In order to show that we can not do better, we observe that if you consider the i-th and $(i+1)$-th elements we should make at least one of them zero at the final stage. This implies that the total amount of gold coins that we spend on the steps in which we reduce the value of at least one of these two elements, is at least i.

  Therefore, one must spend:

  - at least $n-1$ gold coins in the steps that reduce the values of $(n-1)$-th or n-th element,
  - at least $n-4$ gold coins in the steps that reduce the values of $(n-4)$-th or $(n-3)$-th element,
  - $\ldots,$
  - and at least r gold coins in the steps that reduces the values of r-th or $(r+1)$-th element.

  Thus $(n-1)+(n-4)+(n-7)+\cdots+r$ is a lower bound and hence the claim.

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Problem 7.

There are 100 points on the plane, namely $P_1, \ldots, P_{100}$ such that no three points are collinear. Assume for every i and j such that $1 \le i < j-1 \le 98$, the segments P_iP_{i+1}, P_jP_{j+1} do not intersect (i.e., do not share a common point). We say that $\{P_i, P_{i+1}, P_{i+2}\}$ is an *empty* consecutive triple if the triangle $\triangle P_iP_{i+1}P_{i+2}$ does not contain any of the other points. Find the largest number k such that one can always find at least k empty consecutive triples.

Proposed by Morteza Saghafian

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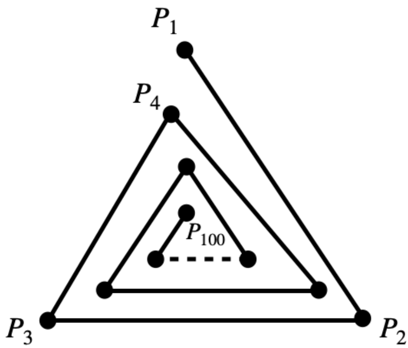
Solution.

- The answer is 1.
- The following example shows that one can not always guarantee 2 empty consecutive triangles. Note that in this example for every $1 \le i < 98$, $\triangle P_iP_{i+1}P_{i+2}$ contains P_{100}.



- Now we prove that there always exists an empty consecutive triangle.

  Claim 1: Assume for some $i \in \{3, \ldots, 100\}$, $P_1P_2 \ldots P_i$ is a polygon that does not intersect itself and does not intersect any of the segments in the path. Then either we can find an empty consecutive triangle or P_{100} is a point that is inside the polygon $P_1P_2 \ldots P_i$.

  Claim 2: There exists some $i \in \{3, \ldots, 100\}$ such that $P_1P_2 \ldots P_i$ is a polygon that does not intersect itself and does not intersect any of the segments in the path.

  Let us assume there are no empty consecutive triangles. By Claims 1 and 2, we can find some $3 \le i < 100$ such that $P_1 \ldots P_i$ is a polygon that satisfies the conditions of Claim 1 and the polygon contains P_{100}. However, by a symmetric argument, we can find some $2 \le j < 99$ such that $P_{100} \ldots P_j$ is a polygon that satisfies the conditions of Claim 1 and the polygon contains P_1. However $P_1 \ldots P_i$ and $P_{100} \ldots P_j$ can not exist at the same time.

Proof of Claim 1: Case 1) We first consider the case that $P_1P_2\dots P_i$ does not contain any of the points in $\{P_{i+1},\dots,P_{100}\}$. If $i=3$, then $\triangle P_1P_2P_3$ is the desired triangle. Otherwise, in a triangulation of $P_1P_2\dots P_i$ there are at least two triangles such as T_1 and T_2, that contain two sides of the polygon. At least one of the two triangles does not contain the segment P_iP_1 and hence it is an empty consecutive triangle. Case 2) If $P_1P_2\dots P_i$ contains a point in $A=\{P_{i+1},\dots,P_{100}\}$ then all the points in A by the assumptions of the claim and also the assumptions in the problem statement. These cases cover the proof of the claim.

Proof of Claim 2: All we need to show is that there exists a point P_i for some $i\in\{3,\dots,100\}$ such that P_1P_i does not intersect any of the segments in the path. For this, we draw a line ℓ that is initially the line that contains P_1 and P_2. Now we fix P_1 as a point of ℓ and as soon as a point P_i such that $i>2$ becomes part of ℓ. Now clearly P_1P_i does not intersect any segment of the path and this means $P_1\dot{P}_i$ is the desired polygon.

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